

# Mean Field Games with Singular Controls of Bounded Velocity

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## Abstract

This paper studies a class of mean field games (MFGs) with singular controls of bounded velocity. By relaxing the absolute continuity of the control process, it generalizes the MFG framework of Lasry and Lions [36] and Huang, Malhamé, and Caines [29]. It provides a unique solution to the MFG with explicit optimal control policies and establishes the  $\epsilon$ -Nash equilibrium of the corresponding  $N$ -player game. Finally, it analyzes a particular MFG with explicit solutions in a systemic risk model originally formulated by Carmona, Fouque, and Sun [17] in a regular control setting.

## 1 Introduction

Let  $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, P)$  be a probability space with  $W^i = \{W_t^i\}_{0 \leq t \leq T}$  i.i.d. standard Brownian motions in this space, for  $i = 1, 2, \dots, N$ . Fix a finite time  $T$ ,  $(s, x^i) \in [0, T] \times \mathbb{R}$ , and a probability measure  $\mu$ . This paper introduces and analyzes the following class of stochastic games,

$$v^i(s, x^i) = \inf_{\xi^{i+}, \xi^{i-} \in \mathcal{U}} E \left[ \int_s^T (f(x_t^i, \mu_t) dt + g_1(x_t^i) d\xi_t^{i+} + g_2(x_t^i) d\xi_t^{i-}) \right], \quad (1)$$

subject to

$$dx_t^i = b(x_t^i, \mu_t) dt + d\xi_t^{i+} - d\xi_t^{i-} + \sigma dW_t^i, \quad \forall t \in [s, T], \quad x_s^i = x^i, \mu_s = \mu. \quad (2)$$

Here  $(\xi_t^{i+}, \xi_t^{i-})$  is a pair of non-decreasing càdlàg processes in an appropriate admissible control set  $\mathcal{U}$ ,  $\mu_t$  is a probability measure of  $x_t^i$ , and  $f, g_1, g_2$  are functions satisfying some technical assumptions to be specified in Section 2.

This kind of problems belongs to a broad class of stochastic games known as the *mean field games* (MFGs). The theoretical development of MFGs is led by the pioneering work of [36] and [29], who studied stochastic games of a large population with small interactions. MFG avoids directly analyzing the notoriously hard  $N$ -player stochastic games when  $N$  is large. Instead, it approximates the dynamics and the objective function under the notion of population's probability distribution flows, a.k.a., mean information processes. (This idea can be traced to physics on weakly interacting particles.) As such, MFG leads to an

elegant and analytically feasible framework to approximate the Nash equilibrium (equilibria) of  $N$ -player stochastic games. The literature on MFG is expanding rapidly. Some recent works include that of Fisher [19], who connects symmetric  $N$ -player games with MFGs, Lacker [35], who analyzes MFGs with controlled martingale problems, Carmona, Delarue, and Lacker [15], who add common noise to MFGs, Nutz [40], who studies MFG of optimal stopping problems with common noise.

**Our work with singular controls.** Most research on MFG theory focuses on regular controls where controls are absolutely continuous and the controlled processes are Lipschitz continuous. In practice, controls are not necessarily absolutely continuous and/or the control rate might be constrained. These types of controls are called *singular controls* or impulse controls depending on the degree of discontinuity of the control. Generally, singular and impulse controls are much harder to analyze. For instance, studying singular controls involves analyzing fully nonlinear PDEs with additional gradient constraints, an important and difficult subject in PDE theory especially in terms of the regularity property. On the other hand, the subject of singular controls has fascinated control theorists, with its distinct “bang-bang” type control policy (Beneš, Shepp, and Witsenhausen [5]) and its connection to optimal stopping and switching (Karatzas and Shreve [32, 33], Boetius [7], Guo and Tomecek [27]).

Our paper studies MFGs with singular control with a bounded velocity for which the controlled process is no longer Lipschitz continuous. For a class of MFGs in the form of Eqn. (5), it shows that under appropriate technical conditions,

- the MFG admits a unique optimal control, and
- the value function of the MFG is an  $\epsilon$ -Nash equilibrium to the corresponding  $N$ -player game, with  $\epsilon = O\left(\frac{1}{\sqrt{N}}\right)$ .

These results are analogous to those for MFGs with regular controls. Furthermore, our paper provides an MFG with singular control with an *explicit* analytic solution. This case study illustrates a curious connection between MFGs with and without common noise, under some “symmetric” problem structure.

**Solution approach.** There are several solution approaches for MFGs. The PDEs/control approach of [36, 12, 29] essentially consists of three steps. The first step is to fix a deterministic mean information process and analyze the corresponding stochastic control problem. Given the solution to the optimal control, the second step is to analyze the optimal controlled process, i.e., the McKean–Vlasov stochastic differential equation (SDE). The third step is to show that the iteration of the first two steps converge, leading to a fixed point solution of the MFG. In this approach, the MFG is essentially analyzed by studying two coupled PDEs, the backward Hamilton–Jacobi–Bellman (HJB) equation and the forward McKean–Vlasov SDE. Buckdahn et al. [8, 9] and Carmona, Delarue, and Lacker [13, 14, 16] propose alternative probabilistic approaches to directly analyze the combined (forward-)backward stochastic differential equations. Recently, Pham and Wei [41, 42] suggest using the stochastic McKean–Vlasov equation and the dynamic programming principle to solve the MFG.

Our solution approach is built on the PDE/control methodology of [36] and [29]. However, the analysis is more difficult for both the HJB equation and the McKean–Vlasov SDEs: not only the HJB equation is with additional state constraints, but also the optimal controlled process is no longer Lipschitz continuous. Our analysis technique is inspired by the work of El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [18] for reflected BSDEs. The key element is to impose the rationality of players. Mathematically, it means that the control is non-increasing with respect to the player’s current state. Intuitively, it says that the better off the state of the individual player, the less likely the player exercises controls (in order to minimize cost).

**Related work.** Some early work on MFG with singular controls includes Zhang [48] and Hu, Oksendal, and Sulem [28]. Both establish the stochastic maximal principle while the latter also proves the existence of optimal control policies for a class of MFGs with singular controls. The work of Fu and Horst [21] adopts the notion of relaxed controls to prove the existence of the solution of MFG with singular controls. Their problem setting and solution approach, however, are different from ours. In addition, our paper establishes both the uniqueness and existence of the solution for MFGs, with explicit structures for the optimal control.

In most MFGs with regular controls, a linear-quadratic problem structure is necessary for the explicit solution (Bardi [1], Bardi and Priuli [2], Bensoussan et al. [6], and [17]). Our paper provides explicit optimal control policies in a general singular control setting; it analyzes a particular MFG with closed-form solutions where the dynamics are linear and the cost structure is not necessarily quadratic (see Remark 1); it also illustrates in this MFG how to explore the particular problem structure to reduce MFGs with common noise to MFGs without common noise, by a simple conditioning trick. A similar trick has been used in [40] for MFG with optimal stopping problems. Earlier theoretical development with common noise can be found in the economic literature; see the papers by Sun [44] and Sun and Zhang [45]. Recently, [15] provides a systematic study for MFGs with common noise in a regular control setting, establishing the existence and uniqueness of both weak and strong solutions.

MFGs are suitable for analyzing many practical problems. Earlier examples include Carmona, Fouque, and Sun [17] and Garnier, Papanicolaou, and Yang [22] for systemic risks, Lasry, Lions, and Guéant [37] for the growth theory in economics, Jaimungal and Nourian [30] and Lachapelle, Lasry, Lehalle, and Lions [34] for high frequency trading, Bauso, Tembine, and Basar [3] and Guéant, Lasry, and Lions [25] for management of non-renewable resources, Caffarelli, Markowich, and Pietschmann [11] for price formation models, Gomes, Mohr, and Souza [23] and Guéant [24] for social network dynamics, and Manjrekar, Ramaswamy, and Shakkottai [39], Wiecek, Altman, and Ghosh [47], and Bayraktar, Budhiraja, and Cohen [4] for queuing systems.

**Outline of the paper.** The paper is organized as follows. Section 2 defines the MFG with singular controls of bounded velocity, and presents the main results regarding the existence and uniqueness of the solution to the MFG, as well as its  $\epsilon$ -Nash equilibrium to the  $N$ -player game. Section 3 provides detailed proofs and Section 4 analyzes a MFG in a systemic risk

model with explicit solutions.

## 2 Problem Formulations and Main Results

Before we define precisely the MFG (1), let us first introduce some notation.

**Notation.** Throughout the paper, we will use the following notation, unless otherwise specified.

- $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, P)$  is a probability space and  $W^i = \{W_t^i\}_{0 \leq t \leq T}$  are i.i.d. standard Brownian motions in this space for  $i = 1, 2, \dots, N$ ;
- $\mathcal{P}(\Omega)$  is the set of all probability measures on  $\Omega$ ;
- $\mathcal{P}_p(\Omega)$  is the set of all probability measures of  $p$  order on  $\Omega$  such that  $\mu \in \mathcal{P}_p(\Omega)$  if  $(\int |x|^p d\mu(x))^{\frac{1}{p}} < \infty$ ;
- $\mathcal{M}_{[0,T]} \subset \mathcal{P}(\mathcal{C}([0,T] : \mathbb{R}))$  is a class of flows of probability measures  $\{\mu_t\}$  on  $[0, T]$  and contains all  $\{\mu_t\}$  for which there exists  $\beta \in (0, 1]$  so that for any bounded and Lipschitz continuous function  $\psi$  on  $\mathbb{R}$ ,

$$\left| \int_{\mathbb{R}} \psi(y) \mu_t(dy) - \int_{\mathbb{R}} \psi(y) \mu_{t'}(dy) \right| \leq c_1 |t - t'|^\beta,$$

for any  $t, t' \in [0, T]$ ;

- $D^p$  is the  $p$ th order Wasserstein metric on  $\mathcal{P}_p(\mathbb{R})$  between two probability measures  $\mu$  and  $\mu'$ , defined as  $D^p(\mu, \mu') = \inf_{\tilde{\mu}} \left( \int |y - y'|^p \tilde{\mu}(dy, dy') \right)^{\frac{1}{p}}$ , where  $\tilde{\mu}$  is a coupling of  $\mu$  and  $\mu'$ ;
- $\mathcal{L}\psi(x) = b(x)\partial_x\psi(x) + \sigma(x)\partial_{xx}\psi(x)$  for any stochastic process  $dx_t = b(x_t)dt + \sigma(x_t)dW_t$  and any  $\psi(x) \in \mathcal{C}^2$ ;
- $Lip(f)$  is a Lipschitz coefficient of  $f$  for any Lipschitz function  $f$ . That is,  $|f(x) - f(y)| \leq Lip(f)|x - y|$  for all  $x, y \in \mathbb{R}$ ;
- A function  $f$  is said to satisfy a polynomial growth condition if  $f(x) \leq c(x^k + 1)$  for some constants  $c, k$ , for all  $x$ .

Next, let us take a look at the corresponding  $N$ -player game.

**$N$ -player games with singular controls.** Fix a finite time  $T$  and denote  $\{x_t^i\}_{0 \leq t \leq T}$  as the state process in  $\mathbb{R}$  for the  $i$ th player ( $i = 1, 2, \dots, N$ ), with  $x_s^i = x^i$  starting from time  $s \in [0, T]$ . Now assume that the dynamics of  $x_t^i$  follows, for  $t \in [s, T]$ ,

$$dx_t^i = \frac{1}{N} \sum_{j=1}^N b_0(x_t^i, x_t^j) dt + \sigma dW_t^i + d\xi_t^i, \quad x_s^i = x^i,$$

for some appropriate function  $b_0 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and a positive constant  $\sigma$ . Here  $\{\xi_t^i\}_{0 \leq t \leq T}$  is the control by the  $i$ th player with  $i = 1, 2, \dots, N$ , assumed to be a càdlàg type and of a finite variation with  $\xi_s^i = 0$ .

By the Lebesgue decomposition theorem, a finite variation process  $\{\xi_t^i\}_{0 \leq t \leq T}$  can be decomposed into two nondecreasing processes  $\{\xi_t^{i+}\}_{0 \leq t \leq T}$ ,  $\{\xi_t^{i-}\}_{0 \leq t \leq T}$  such that  $\xi_t^i = \xi_t^{i+} - \xi_t^{i-}$  with  $\xi_s^{i+} = \xi_s^{i-} = 0$ . Therefore the dynamics of  $x_t^i$  can be rewritten as

$$dx_t^i = \frac{1}{N} \sum_{j=1}^N b_0(x_t^i, x_t^j) dt + \sigma dW_t^i + d\xi_t^{i+} - d\xi_t^{i-}, \quad x_s^i = x^i. \quad (3)$$

The objective of the  $i$ th player is to minimize a cost function  $J_N^i(s, x, \xi^{i+}, \xi^{i-}) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  given by

$$J_N^i(s, x^i, \xi^{i+}, \xi^{i-}) = E \left[ \int_s^T \frac{1}{N} \sum_{j=1}^N f_0(x_t^i, x_t^j) dt + g_1(x_t^i) d\xi_t^{i+} + g_2(x_t^i) d\xi_t^{i-} \right],$$

for some functions  $f_0 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ , and over an appropriate admissible control set  $\mathcal{U}_0$ .

That is, the  $i$ th player's control problem (if well-defined) is to solve

$$\inf_{\xi^{i+}, \xi^{i-} \in \mathcal{U}_0} J_N^i(s, x^i, \xi^{i+}, \xi^{i-}) = \inf_{\xi^{i+}, \xi^{i-} \in \mathcal{U}_0} E \left[ \int_s^T \frac{1}{N} \sum_{j=1}^N f_0(x_t^i, x_t^j) dt + g_1(x_t^i) d\xi_t^{i+} + g_2(x_t^i) d\xi_t^{i-} \right], \quad (4)$$

subject to Eqn. (3).

Note that in this  $N$ -player game, both the drift term in Eqn. (3) for the dynamics and the first term in Eqn. (4) for the objective function are affected by both the local information (i.e., the state of the  $i$ th player itself) and the global information (i.e., the states of other players). In general, this type of stochastic game is difficult to analyze: although the work of Uchida [46] shows the existence of Nash equilibrium for such an  $N$ -player game, finding the solution for the  $N$ -player game is in general intractable.

**A heuristic derivation to the MFG formulation.** Assume that all  $N$  players are identical and interchangeable, and let  $\mu_t$  be the probability measure of  $x_t^i$  for  $i = 1, 2, \dots, N$ . Then under appropriate technical conditions, one can approximate via  $\mu_t$ , according to SLLN, the drift function and cost function of the  $i$ th player game when  $N \rightarrow \infty$ , so that

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^N b_0(x_t^i, x_t^j) &\rightarrow \int b_0(x_t^i, y) \mu_t(dy) = b(x_t^i, \mu_t), \\ \frac{1}{N} \sum_{j=1}^N f_0(x_t^i, x_t^j) &\rightarrow \int f_0(x_t^i, y) \mu_t(dy) = f(x_t^i, \mu_t). \end{aligned}$$

This leads to an MFG formulation of (4) as in Eqn. (1). Note that now the drift term in the dynamics and the objective function rely only on the local information  $x_t^i$  and the aggregated mean information  $\mu_t$ .

Assume that the controls  $\xi_t^{i+}, \xi_t^{i-}$  are with bounded velocity so that  $d\xi_t^{i+} = \dot{\xi}_t^{i+} dt$  and  $d\xi_t^{i-} = \dot{\xi}_t^{i-} dt$  with  $0 \leq \dot{\xi}_t^{i+}, \dot{\xi}_t^{i-} \leq \theta$  for a constant  $\theta > 0$ . Then the MFG problem of Eqn. (1) can be defined precisely as

$$\begin{aligned} v^i(s, x^i) &= \inf_{\xi^{i+}, \xi^{i-} \in \mathcal{U}} J_\infty^i(s, x^i, \dot{\xi}^{i+}, \dot{\xi}^{i-}) \\ &= \inf_{\xi^{i+}, \xi^{i-} \in \mathcal{U}} E \left[ \int_s^T \left( f(x_t^i, \mu_t) + g_1(x_t^i) \dot{\xi}_t^{i+} + g_2(x_t^i) \dot{\xi}_t^{i-} \right) dt \right], \quad (5) \\ \text{subject to } dx_t^i &= \left( b(x_t^i, \mu_t) + \dot{\xi}_t^{i+} - \dot{\xi}_t^{i-} \right) dt + \sigma dW_t^i, \quad x_s^i = x^i, \mu_s = \mu. \end{aligned}$$

Here the admissible control set is

$$\mathcal{U} = \{ \{ \dot{\xi}_t \} | \{ \xi_t \} \text{ is } \mathcal{F}_t\text{-progressively measurable, nondecreasing and } \xi_0 = 0, \dot{\xi}_t \in U = [0, \theta] \},$$

$\{ \mu_t \}_{0 \leq t \leq T} \in \mathcal{M}_{[0, T]}$ ,  $\mu_t$  is a probability measure of  $x_t^i$ ,  $b, f : \mathbb{R} \times \mathcal{M}_{[0, T]} \rightarrow \mathbb{R}$  and  $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$  are functions satisfying the following assumptions.

### Assumptions.

- (A1)  $b_0(x, y)$ ,  $f_0(x, y)$ ,  $g_1(x)$ , and  $g_2(x)$  (hence  $b(x, y)$  and  $f(x, y)$ ) are bounded and Lipschitz continuous in  $x$  and  $y$ ;
- (A2)  $b_0(x, y)$ ,  $f_0(x, y)$ ,  $g_1(x)$ , and  $g_2(x)$  (hence  $b(x, y)$  and  $f(x, y)$ ) have first and second order derivatives with respect to  $x$ , and derivatives are uniformly continuous and bounded in  $x$  and Lipschitz continuous in  $y$ ;
- (A3)  $-g_1(x) \leq g_2(x)$  for any  $x \in \mathbb{R}$ ;
- (A4) (Rationality of players) For any control functions  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $(x - y)(\varphi(x) - \varphi(y)) \leq 0$ ;
- (A5) (Feedback regularity condition) Let  $\varphi, \tilde{\varphi}$  be two optimal control functions under given two flows of probability measures  $\{ \mu_t \}$  and  $\{ \tilde{\mu}_t \}$ . Then, there exists a constant  $d_2 > 0$  such that

$$E \left[ \int_0^t |\varphi(x_s | \{ \mu_s \}) - \tilde{\varphi}(\tilde{x}_s | \{ \tilde{\mu}_s \})| ds \right] \leq d_2 D^1(\mu_t, \tilde{\mu}_t), \quad (6)$$

where  $x_t, \tilde{x}_t$  are corresponding state processes from  $x_0 = \tilde{x}_0 = x$  under  $\varphi, \tilde{\varphi}, \mu$ , and  $\tilde{\mu}$ .

While (A3) ensures the finiteness of the value function, (A5) and in particular the constant  $d_2$  is critical to ensure the uniqueness of fixed point solution for the MFG, just as in the regular control setting in [29].

Throughout the paper, we will restrict ourselves to controls of Markov types.

**Solution approach and main results.** Our solution approach is in the spirit of [36, 29], and consists of three steps. The first step is to analyze a stochastic control problem under a fixed flow of probability measures  $\{ \mu_t \}$ . If such a control problem has a unique optimal control, denoted as  $\dot{\xi}_t^* dt = \varphi(x_t | \{ \mu_t \}) dt$ , then one can proceed to define a mapping  $\Gamma_1$  from the class of probability measure flows  $\mathcal{M}_{[0, T]}$  to the space of measurable functions so

that  $\Gamma_1(\{\mu_t\}) = \varphi(x|\{\mu_t\})$ . The second step is to analyze the optimal controlled process, the McKean–Vlasov SDE, given the optimal control function  $\varphi$ . If this SDE allows for a unique flow of probability measures solution in  $\mathcal{M}_{[0,T]}$ , denoted as  $\{\mu(\varphi)_t\}_{0 \leq t \leq T}$ , then one can define another mapping  $\Gamma_2$  from the space of measurable functions to  $\mathcal{M}_{[0,T]}$  so that  $\Gamma_2(\varphi) = \{\mu(\varphi)_t\}$ . The final step is to check if  $\Gamma_1 \circ \Gamma_2$  is a contraction mapping to allow for a fixed point solution, leading to the solution of the MFG.

Let us first recall some basic definitions.

**Definition 1.** A solution of the MFG is defined as a pair of optimal controlled processes  $\xi_t^{i*} = \xi_t^{i*+} - \xi_t^{i*-}$  for  $i = 1, \dots, N$  and a flow of probability measures  $\{\mu_t^*\} \in \mathcal{M}_{[0,T]}$  if they satisfy  $v^i(s, x) = J_\infty^i(s, x, \dot{\xi}_t^{i*+}, \dot{\xi}_t^{i*-})$  for all  $(s, x) \in [0, T] \times \mathbb{R}$  where  $\mu_t^*$  is a probability measure of the optimal controlled process  $x_t^{i*}$  for all  $t \in [0, T]$ .

**Definition 2** ( $\epsilon$ -Nash equilibrium).  $\{\xi_t^{i*+}, \xi_t^{i*-}\}_{i=1}^n$  is called an  $\epsilon$ -Nash equilibrium if for any  $i \in \{1, 2, \dots, n\}$  and any  $\dot{\xi}_t^{i'+}, \dot{\xi}_t^{i'-} \in \mathcal{U}$ ,  $J_N^i(\dot{\xi}_t^{i'+}, \dot{\xi}_t^{i'-}, \xi^{*-i}) \geq J_N^i(\dot{\xi}_t^{i*+}, \dot{\xi}_t^{i*-}, \xi^{*-i}) - \epsilon$ , where  $J_N^i$  is the cost function for the  $i$ th player and  $\xi^{*-i}$  is the control processes  $\{\xi_t^{j*+}, \xi_t^{j*-}\}_{j=1, j \neq i}^n$  by all players except the  $i$ th player.

Next, we have,

**Lemma 1.** Under two fixed optimal control functions  $\varphi$  and  $\tilde{\varphi}$ , consider two McKean–Vlasov equations<sup>1</sup> given by

$$\begin{aligned} x_t &= x + \int_0^t b(x_s, \mu(\varphi)_s) + \varphi(x_s) ds + \int_0^t \sigma dW_s, \\ \tilde{x}_t &= x + \int_0^t b(\tilde{x}_s, \mu(\tilde{\varphi})_s) + \tilde{\varphi}(\tilde{x}_s) ds + \int_0^t \sigma dW_s. \end{aligned}$$

where  $\mu(\varphi)_t$  and  $\mu(\tilde{\varphi})_t$  are probability measures of  $x_t$  and  $\tilde{x}_t$ . Then, there exists  $d_1 > 0$  satisfying

$$D^1(\mu(\varphi)_t, \mu(\tilde{\varphi})_t) \leq d_1 E \left[ \int_0^t |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| ds \right],$$

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<sup>1</sup>We will see from Proposition 2 in Section 3 that the above McKean–Vlasov equations indeed have unique solutions  $(x_t, \mu(\varphi)_t)$  and  $(\tilde{x}_t, \mu(\tilde{\varphi})_t)$ , respectively.

*Proof.* Let  $\varphi$  and  $\tilde{\varphi}$  be two given control functions, then,

$$\begin{aligned}
|x_t - \tilde{x}_t| &\leq \int_0^t (|b(x_s, \mu(\varphi)_s) - b(\tilde{x}_s, \mu(\tilde{\varphi})_s)| + |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)|) ds \\
&\leq \int_0^t \left( \left| \int_{\mathbb{R}} b_0(x_s, y) \mu(\varphi)_s(dy) - \int_{\mathbb{R}} b_0(\tilde{x}_s, \tilde{y}) \mu(\tilde{\varphi})_s(d\tilde{y}) \right| + |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| \right) ds \\
&\leq \int_0^t \left( \left| \int_{\mathbb{R} \times \mathbb{R}} [b_0(x_s, y) - b_0(\tilde{x}_s, \tilde{y})] (\mu(\varphi) \times \mu(\tilde{\varphi}))_s(dy, d\tilde{y}) \right| + |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| \right) ds \\
&\leq \int_0^t \left( \int_{\mathbb{R} \times \mathbb{R}} |b_0(x_s, y) - b_0(\tilde{x}_s, y) + b_0(\tilde{x}_s, y) - b_0(\tilde{x}_s, \tilde{y})| (\mu(\varphi) \times \mu(\tilde{\varphi}))_s(dy, d\tilde{y}) \right. \\
&\quad \left. + |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| \right) ds \\
&\leq \int_0^t \left( \int_{\mathbb{R} \times \mathbb{R}} Lip(b_0)(|x_s - \tilde{x}_s| + |y - \tilde{y}|) (\mu(\varphi) \times \mu(\tilde{\varphi}))_s(dy, d\tilde{y}) + |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| \right) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|x_t - \tilde{x}_t| &\leq Lip(b_0) \int_0^t |x_s - \tilde{x}_s| ds \\
&\quad + Lip(b_0) \int_0^t D^1(\mu(\varphi)_s, \mu(\tilde{\varphi})_s) ds + \int_0^t |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| ds.
\end{aligned}$$

By the Gronwall's inequality, there exists some  $d_3 > 0$  such that

$$|x_t - \tilde{x}_t| \leq d_3 \left( Lip(b_0) \int_0^t D^1(\mu(\varphi)_s, \mu(\tilde{\varphi})_s) ds + \int_0^t |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| ds \right).$$

and

$$D^1(\mu(\varphi)_t, \mu(\tilde{\varphi})_t) \leq d_3 \left( Lip(b_0) \int_0^t D^1(\mu(\varphi)_s, \mu(\tilde{\varphi})_s) ds + \int_0^t |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| ds \right).$$

By the Gronwall's inequality again, there exists  $d_1 > 0$  such that

$$D^1(\mu(\varphi)_t, \mu(\tilde{\varphi})_t) \leq d_1 E \left[ \int_0^t |\varphi(x_s) - \tilde{\varphi}(\tilde{x}_s)| ds \right]. \quad (7)$$

This completes the proof.  $\square$

Now, we are ready to state the main results of the paper.

**Theorem 1.** Assume (A1)–(A5) and  $d_1 d_2 < 1$  with  $d_1, d_2$  given in Eqns (7) and (6) respectively. Then there exists a unique solution  $(\xi_t^{i*}, \{\mu_t^{i*}\})$  to the MFG (5). Moreover, the corresponding value function  $v^i$  for the MFG (5) is a function in  $C^{1,2}([0, T] \times \mathbb{R})$ , of a polynomial growth, with the optimal control given by

$$\varphi(x | \{\mu_t^{i*}\}) = \dot{\xi}_t^{i*+} - \dot{\xi}_t^{i*-} = \begin{cases} \theta & \text{if } \partial_x v^i \leq -g_1(x), \\ 0 & \text{if } -g_1(x) \leq \partial_x v^i \leq g_2(x), \\ -\theta & \text{if } g_2(x) \leq \partial_x v^i. \end{cases}$$



**Theorem 2.** Assuming (A1)–(A5), the optimal control to the mean field game (5) is an  $\epsilon$ -Nash equilibrium to the corresponding  $N$ -player game (8), with  $\epsilon = O(\frac{1}{\sqrt{N}})$ . Here the corresponding  $N$ -player game is, for any  $(s, x^i) \in [0, T] \times \mathbb{R}$ ,

$$\begin{aligned} \inf_{\dot{\xi}^{i+}, \dot{\xi}^{i-} \in \mathcal{U}} J_N^i(s, x^i, \dot{\xi}^{i+}, \dot{\xi}^{i-}) &= \inf_{\dot{\xi}^{i+}, \dot{\xi}^{i-} \in \mathcal{U}} E \int_s^T \left( \frac{1}{N} \sum_{j=1}^N f_0(x_t^i, x_t^j) + g_1(x_t^i) \dot{\xi}_t^{i+} + g_2(x_t^i) \dot{\xi}_t^{i-} \right) dt, \\ \text{subject to } dx_t^i &= \left( \frac{1}{N} \sum_{j=1}^N b_0(x_t^i, x_t^j) + \dot{\xi}_t^{i+} - \dot{\xi}_t^{i-} \right) dt + \sigma dW_t^i, \quad x_s^i = x^i. \end{aligned} \quad (8)$$

## 3 Proofs

### 3.1 Proof of Theorem 1

#### 3.1.1 The Stochastic Control Problem

Let  $\{\mu_t\}$  be a fixed exogenous flow of probability measures. Then, (5) is the following control problem for  $i$ th player,

$$v^i(s, x^i) = \inf_{\dot{\xi}^{i+}, \dot{\xi}^{i-} \in \mathcal{U}} E \left[ \int_s^T \left( f(x_t^i, \mu_t) + g_1(x_t^i) \dot{\xi}_t^{i+} + g_2(x_t^i) \dot{\xi}_t^{i-} \right) dt \right], \quad (9)$$

subject to

$$dx_t^i = \left( b(x_t^i, \mu_t) + \dot{\xi}_t^{i+} - \dot{\xi}_t^{i-} \right) dt + \sigma dW_t^i, \quad x_s^i = x^i, \mu_s = \mu.$$

This is a classical stochastic control problem, and the corresponding HJB equation with the terminal condition is given by <sup>2</sup>

$$\begin{aligned} -\partial_t v &= \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0, \theta]} \left\{ \left( b(x, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x v + \left( f(x, \mu) + g_1(x) \dot{\xi}^+ + g_2(x) \dot{\xi}^- \right) \right\} + \frac{\sigma^2}{2} \partial_{xx} v \\ &= \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0, \theta]} \left\{ (\partial_x v + g_1(x)) \dot{\xi}^+ + (-\partial_x v + g_2(x)) \dot{\xi}^- \right\} + b(x, \mu) \partial_x v + f(x, \mu) + \frac{\sigma^2}{2} \partial_{xx} v \\ &= \min \{ (\partial_x v + g_1(x)) \theta, (-\partial_x v + g_2(x)) \theta, 0 \} + b(x, \mu) \partial_x v + f(x, \mu) + \frac{\sigma^2}{2} \partial_{xx} v, \\ &\text{with } v(T, x) = 0, \quad \forall x \in \mathbb{R}. \end{aligned} \quad (10)$$

The existence and uniqueness of a  $C^{1,2}([0, T] \times \mathbb{R})$  solution to (10) is clear by Theorem 6.2. in [20]. Moreover, we can show that such a solution to (10) is the value function of (9).

Before establishing this result, let us recall the viscosity solution to (10).

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<sup>2</sup>Because all players are identical, we will for the remaining part of this section omit the superscript  $i$  for simplicity.

**Definition 3.**  $\hat{v}$  is called a viscosity solution to (10) if  $\hat{v}$  is both a viscosity supersolution and a viscosity subsolution, with the following definitions,

(i) viscosity supersolution: for any  $(t_0, x_0) \in [0, T] \times \mathbb{R}$  and any  $\vartheta \in \mathcal{C}^{1,2}$ , if  $(t_0, x_0)$  is a local minimum of  $\hat{v} - \vartheta$  with  $\hat{v}(t_0, x_0) - \vartheta(t_0, x_0) = 0$ , then

$$\begin{aligned} -\partial_t \vartheta(t_0, x_0) - \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0, \theta]} \left\{ \left( b(x_0, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x \vartheta(t_0, x_0) + \left( f(x_0, \mu) + g_1(x_0) \dot{\xi}^+ + g_2(x_0) \dot{\xi}^- \right) \right\} \\ - \frac{\sigma^2}{2} \partial_{xx} \vartheta(t_0, x_0) \geq 0, \quad \text{and} \quad \vartheta(T, x_0) \geq 0; \end{aligned}$$

(ii) viscosity subsolution: for any  $(t_0, x_0) \in [0, T] \times \mathbb{R}$  and any  $\vartheta \in \mathcal{C}^{1,2}$ , if  $(t_0, x_0)$  is a local maximum of  $\hat{v} - \vartheta$  with  $\hat{v}(t_0, x_0) - \vartheta(t_0, x_0) = 0$ , then

$$\begin{aligned} -\partial_t \vartheta(t_0, x_0) - \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0, \theta]} \left\{ \left( b(x_0, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x \vartheta(t_0, x_0) + \left( f(x_0, \mu) + g_1(x_0) \dot{\xi}^+ + g_2(x_0) \dot{\xi}^- \right) \right\} \\ - \frac{\sigma^2}{2} \partial_{xx} \vartheta(t_0, x_0) \leq 0, \quad \text{and} \quad \vartheta(T, x_0) \leq 0. \end{aligned}$$

**Proposition 1.** Assume a fixed  $\{\mu_t\}$  in  $\mathcal{M}_{[0, T]}$  for  $0 \leq t \leq T$ . Under Assumptions (A1)-(A4), the HJB Eqn. (10) with a terminal condition  $v(T, x) = 0$  for any  $x \in \mathbb{R}$  has a unique solution  $v$  in  $C^{1,2}([0, T] \times \mathbb{R})$ , of a polynomial growth. Furthermore, the solution is the value function to the problem (9), with the optimal control given by

$$\varphi(x|\{\mu_t\}) = \dot{\xi}^+ - \dot{\xi}^- = \begin{cases} \theta & \text{if } \partial_x v \leq -g_1(x), \\ 0 & \text{if } -g_1(x) \leq \partial_x v \leq g_2(x), \\ -\theta & \text{if } g_2(x) \leq \partial_x v. \end{cases}$$

*Proof.* Let  $w$  be the  $C^{1,2}([0, T] \times \mathbb{R})$  solution with polynomial growth to (10) and  $v$  is the value function of (9).

First,  $w$  is a viscosity subsolution to (10). That is, for any  $(s, x) \in [0, T] \times \mathbb{R}$ ,  $w(T, x) \leq 0$  and

$$-\partial_t w - \inf_{\dot{\xi}^+, \dot{\xi}^- \in [0, \theta]} \left\{ \left( b(x, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x w + \left( f(x, \mu) + g_1(x) \dot{\xi}^+ + g_2(x) \dot{\xi}^- \right) \right\} - \frac{\sigma^2}{2} \partial_{xx} w \leq 0.$$

On one hand, for any  $\dot{\xi}_t^+, \dot{\xi}_t^- \in \mathcal{U}$ , let  $x_t$  be a controlled process with  $\dot{\xi}_t^+, \dot{\xi}_t^-$ . Let  $\tau_n$  be stopping times a.s. finite for each  $n$ , with  $\tau_n \rightarrow T$  as  $n \rightarrow \infty$ . Then, by the Itô's formula on  $w(s, x)$ ,

$$\begin{aligned} E[w(T \wedge \tau_n, x_{T \wedge \tau_n})] \\ = w(s, x) + E \left[ \int_s^{T \wedge \tau_n} \partial_t w(t, x_t) + (b(x_t, \mu_t) + (\dot{\xi}_t^+ - \dot{\xi}_t^-)) \partial_x w(t, x_t) + \frac{\sigma^2}{2} \partial_{xx} w(t, x_t) dt \right] \\ \geq w(s, x) - E \left[ \int_s^{T \wedge \tau_n} f(x_t, \mu_t) + g_1(x_t) \dot{\xi}_t^+ + g_2(x_t) \dot{\xi}_t^- dt \right]. \end{aligned}$$

Taking  $n \rightarrow \infty$ ,

$$0 \geq E[w(T, x_T)] \geq w(s, x) - E \left[ \int_s^T f(x_t, \mu_t) + g_1(x_t) \dot{\xi}_t^+ + g_2(x_t) \dot{\xi}_t^- dt \right].$$

Hence, for any  $\dot{\xi}_t^+, \dot{\xi}_t^- \in \mathcal{U}$ ,

$$E \left[ \int_s^T f(x_t, \mu_t) + g_1(x_t) \dot{\xi}_t^+ + g_2(x_t) \dot{\xi}_t^- dt \right] \geq w(s, x).$$

Therefore,

$$v(s, x) = \inf_{\dot{\xi}_t^+, \dot{\xi}_t^- \in \mathcal{U}} E \left[ \int_s^T f(x_t, \mu_t) + g_1(x_t) \dot{\xi}_t^+ + g_2(x_t) \dot{\xi}_t^- dt \right] \geq w(s, x).$$

On the other hand, let  $\dot{\xi}_t^{*+}, \dot{\xi}_t^{*-} \in \mathcal{U}$  be a minimizer of the Hamiltonian

$$\inf_{\dot{\xi}^+, \dot{\xi}^- \in [0, \theta]} \left\{ \left( b(x, \mu) + (\dot{\xi}^+ - \dot{\xi}^-) \right) \partial_x w + \left( f(x, \mu) + g_1(x) \dot{\xi}^+ + g_2(x) \dot{\xi}^- \right) \right\}.$$

Then, since  $w(s, x)$  is the solution to (10),  $w(T, x) = 0$  for any  $x \in \mathbb{R}$  and with controls  $\dot{\xi}_t^{*+}, \dot{\xi}_t^{*-}$

$$-\partial_t w - \left\{ \left( b(x, \mu) + (\dot{\xi}^{*+} - \dot{\xi}^{*-}) \right) \partial_x w + \left( f(x, \mu) + g_1(x) \dot{\xi}^{*+} + g_2(x) \dot{\xi}^{*-} \right) \right\} - \frac{\sigma^2}{2} \partial_{xx} w = 0.$$

Let  $x_t^*$  be the controlled process with controls  $\dot{\xi}_t^{*+}, \dot{\xi}_t^{*-}$  and  $\tau_n$  be stopping times a.s. finite for each  $n$ , with  $\tau_n \rightarrow T$  as  $n \rightarrow \infty$ . Then, applying the Itô's formula to  $w(t, x)$ ,

$$\begin{aligned} & E[w(T \wedge \tau_n, x_{T \wedge \tau_n})] \\ &= w(s, x) + E \left[ \int_s^{T \wedge \tau_n} \left( \partial_t w(t, x_t) + (b(x_t, \mu_t) + (\dot{\xi}_t^{*+} - \dot{\xi}_t^{*-})) \partial_x w(t, x_t) + \frac{\sigma^2}{2} \partial_{xx} w(t, x_t) \right) dt \right] \\ &= w(s, x) - E \left[ \int_s^{T \wedge \tau_n} \left( f(x_t, \mu_t) + g_1(x_t) \dot{\xi}_t^{*+} + g_2(x_t) \dot{\xi}_t^{*-} \right) dt \right]. \end{aligned}$$

As  $n \rightarrow \infty$ ,

$$\begin{aligned} 0 &= E[w(T, x_T)] = w(s, x) - E \left[ \int_s^T f(x_t, \mu_t) + g_1(x_t) \dot{\xi}_t^{*+} + g_2(x_t) \dot{\xi}_t^{*-} dt \right] \\ &\leq w(s, x) - v(s, x). \end{aligned}$$

Hence,  $v(s, x) \leq w(s, x)$ .

Combined,  $v(s, x) = w(s, x)$  and  $\dot{\xi}_t^{*+}, \dot{\xi}_t^{*-}$  are the optimal controls.  $\square$

Now, one can define  $\Gamma_1$  from the class of flows of probability measures  $\mathcal{M}_{[0, T]}$  to the space of measurable functions so that  $\Gamma_1(\{\mu_t\}) = \varphi(x|\{\mu_t\})$  which is an optimal control function under fixed  $\{\mu_t\}$ .

### 3.1.2 The McKean–Vlasov Equation

Now with a bounded and measurable  $\varphi(x|\{\mu_t\})$  as an optimal control function for a fixed  $\{\mu_t\}$ , the dynamics of the corresponding controlled process  $x_t^i$  follows

$$dx_t^i = (b(x_t^i, \mu(\varphi)_t) + \varphi(x_t^i|\{\mu_t\})) dt + \sigma dW_t^i, \quad x_0^i = x^i \quad (11)$$

where  $\mu(\varphi)_t$  is a probability measure for  $x_t^i$ .

**Proposition 2.** *The McKean–Vlasov type SDE (11) has a unique solution  $(x_t, \mu(\varphi)_t)$ , with  $\{\mu(\varphi)_t\}_{0 \leq t \leq T}$  in  $\mathcal{M}_{[0,T]}$ .*

*Proof.* Take any  $\{\mu_t^1\}, \{\tilde{\mu}_t^1\} \in \mathcal{M}_{[0,T]}$ , and let  $\{x_t^i\}, \{\tilde{x}_t^i\}$  be two processes with  $x_0^i = \tilde{x}_0^i = x^i$  such that

$$\begin{aligned} dx_s^i &= (b(x_s^i, \mu_s^1) + \varphi(x_s^i | \{\mu_s\}))ds + \sigma dW_s^i \\ d\tilde{x}_s^i &= (b(\tilde{x}_s^i, \tilde{\mu}_s^1) + \varphi(\tilde{x}_s^i | \{\mu_s\}))ds + \sigma dW_s^i. \end{aligned}$$

Then,

$$d(x_s^i - \tilde{x}_s^i) = (b(x_s^i, \mu_s^1) - b(\tilde{x}_s^i, \tilde{\mu}_s^1) + \varphi(x_s^i) - \varphi(\tilde{x}_s^i))ds.$$

By the chain rule and Assumption (A4),

$$\begin{aligned} |x_t^i - \tilde{x}_t^i|^2 &= 2 \int_0^t (b(x_s^i, \mu_s^1) - b(\tilde{x}_s^i, \tilde{\mu}_s^1) + \varphi(x_s^i) - \varphi(\tilde{x}_s^i))(x_s^i - \tilde{x}_s^i)ds \\ &\leq 2 \int_0^t \left| \int_{\mathbb{R}} b_0(x_s^i, y) \mu_s^1(dy) - \int_{\mathbb{R}} b_0(\tilde{x}_s^i, \tilde{y}) \tilde{\mu}_s^1(d\tilde{y}) \right| |x_s^i - \tilde{x}_s^i| + (\varphi(x_s^i) - \varphi(\tilde{x}_s^i))(x_s^i - \tilde{x}_s^i)ds \\ &\leq 2 \int_0^t \left| \int_{\mathbb{R} \times \mathbb{R}} [b_0(x_s^i, y) - b_0(\tilde{x}_s^i, \tilde{y})] (\mu^1 \times \tilde{\mu}^1)_s(dy, d\tilde{y}) \right| |x_s^i - \tilde{x}_s^i| ds \\ &\leq 2 \int_0^t |x_s^i - \tilde{x}_s^i| \int_{\mathbb{R} \times \mathbb{R}} |b_0(x_s^i, y) - b_0(\tilde{x}_s^i, y) + b_0(\tilde{x}_s^i, y) - b_0(\tilde{x}_s^i, \tilde{y})| (\mu^1 \times \tilde{\mu}^1)_s(dy, d\tilde{y}) ds \\ &\leq 2 \int_0^t \int_{\mathbb{R} \times \mathbb{R}} Lip(b_0)(|x_s^i - \tilde{x}_s^i|^2 + |x_s^i - \tilde{x}_s^i||y - \tilde{y}|) (\mu^1 \times \tilde{\mu}^1)_s(dy, d\tilde{y}) ds. \end{aligned}$$

Consequently,

$$\begin{aligned} |x_t^i - \tilde{x}_t^i|^2 &\leq 2Lip(b_0) \int_0^t |x_s^i - \tilde{x}_s^i|^2 ds \\ &\quad + Lip(b_0) \int_0^t \int_{\mathbb{R} \times \mathbb{R}} (|x_s^i - \tilde{x}_s^i|^2 + |y - \tilde{y}|^2) (\mu^1 \times \tilde{\mu}^1)_s(dy, d\tilde{y}) ds \\ &\leq 3Lip(b_0) \int_0^t |x_s^i - \tilde{x}_s^i|^2 dt + Lip(b_0) \int_0^t \int_{\mathbb{R} \times \mathbb{R}} |y - \tilde{y}|^2 (\mu^1 \times \tilde{\mu}^1)_s(dy, d\tilde{y}) ds. \end{aligned}$$

Hence,  $|x_t^i - \tilde{x}_t^i|^2 \leq 3Lip(b_0) \int_0^t |x_s^i - \tilde{x}_s^i|^2 ds + Lip(b_0) \int_0^t (D^2(\mu_s^1, \tilde{\mu}_s^1))^2 ds$ . By the Gronwall's inequality, if  $\phi(\mu^1)$  and  $\phi(\tilde{\mu}^1)$  are the distributions of  $x_t$  and  $\tilde{x}_t$ , respectively, then

$$(D^2(\phi(\mu^1)_t, \phi(\tilde{\mu}^1)_t))^2 \leq c_t \int_0^t (D^2(\mu_s^1, \tilde{\mu}_s^1))^2 ds.$$

As in the proof of Theorem 1.1 in [43], one can generate a Cauchy sequence  $\{\phi^k(\mu^1)\}_{k \geq 0}$  satisfying  $(D^2(\phi^{k+1}(\mu^1)_t, \phi^k(\mu^1)_t))^2 \leq c_t \frac{t^k}{k!} (D^2(\phi(\mu^1)_t, \mu_t^1))^2$ . By the contraction mapping theorem, it has a fixed point. Therefore, (11) has a unique consistent solution  $(x_t, \mu(\varphi)_t)$ .

It remains to see that the solution to (11)  $\{\mu(\varphi)_t\}_{0 \leq t \leq T}$  is in  $\mathcal{M}_{[0,T]}$ . Without loss of generality, suppose  $t' > t$ , and  $x_{t'} = x_t + \int_t^{t'} (b(x_s, \mu(\varphi)_s) + \varphi(x_s | \{\mu_s\}))ds + \int_t^{t'} \sigma dW_s$ . Since  $b(x, \mu(\varphi)_s) = \int b_0(x, y) \mu(\varphi)_s(dy)$  is bounded and  $|\varphi(x_s | \{\mu_s\})| \leq \theta$ , the drift function is bounded. Now, similar to the proof of Lemma 7 in [29],  $\{\mu(\varphi)_t\}_{0 \leq t \leq T}$  is in  $\mathcal{M}_{[0,T]}$ .  $\square$

Consequently, one can define  $\Gamma_2$  from the space of measurable functions to the class of flows of probability measures  $\mathcal{M}_{[0,T]}$  to so that  $\Gamma_2(\varphi) = \{\mu(\varphi)_t\}$  which is an updated mean information probability measure flow under the fixed control function  $\varphi$ .

### 3.1.3 The Fixed Point of Iteration

Now, suppose  $\{\mu_t\}$  and  $\{\tilde{\mu}_t\}$  are two different flows of probability measures, with  $\varphi$  and  $\tilde{\varphi}$  their corresponding optimal control functions respectively. Assume  $x_0^i = \tilde{x}_0^i = x^i$ , then according to Proposition 2, there exist two consistent pairs  $(x_t^i, \mu(\varphi)_t)$  and  $(\tilde{x}_t^i, \mu(\tilde{\varphi})_t)$  such that

$$\begin{aligned} x_t^i &= x^i + \int_0^t (b(x_s^i, \mu(\varphi)_s) + \varphi(x_s^i)) ds + \sigma W_t^i, \\ \tilde{x}_t^i &= x^i + \int_0^t (b(\tilde{x}_s^i, \mu(\tilde{\varphi})_s) + \tilde{\varphi}(\tilde{x}_s^i)) ds + \sigma W_t^i. \end{aligned}$$

Now, Theorem 1 is clear from the feedback regularity condition and Lemma 1, as

$$D^1(\mu(\varphi)_t, \mu(\tilde{\varphi})_t) \leq d_1 d_2 D^1(\mu_t, \tilde{\mu}_t),$$

and there exists a unique fixed point if  $d_1 d_2 < 1$ .

## 3.2 Proof of Theorem 2

There are three steps to show that the fixed point solution of MFG (5) is an  $\epsilon$ -Nash equilibrium of the corresponding  $N$ -player game (8).

First, let  $(\varphi, \{\mu_t\})$  be a fixed point solution to the MFG (5). Then the corresponding stochastic differential equation for the MFG is

$$dx_t^i = \left( \int b_0(x_t^i, y) \mu_t(dy) + \varphi(x_t^i) \right) dt + \sigma dW_t^i. \quad (12)$$

Here,  $\mu_t$  is a distribution of  $x_t^i$  for  $i = 1, 2, \dots, N$ . Meanwhile, the equation for the  $i$ th player from the associated  $N$ -player game with the same control  $\varphi$  is

$$d\hat{x}_t^i = \left( \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, \hat{x}_t^j) + \varphi(\hat{x}_t^i) \right) dt + \sigma dW_t^i, \quad (13)$$

with  $x_0^i = \hat{x}_0^i = x^i$ . We have

**Lemma 2.** *Given equations (12) and (13),  $\sup_{0 \leq t \leq T} |x_t^i - \hat{x}_t^i|^2 = O(\frac{1}{N})$ .*

*Proof.* Let  $\bar{x}_t^i = x_t^i - \hat{x}_t^i$ , then

$$d\bar{x}_t^i = \left( \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, \hat{x}_t^j) + \varphi(x_t^i) - \varphi(\hat{x}_t^i) \right) dt,$$

and

$$d((\bar{x}_t^i)^2) = \left( 2\bar{x}_t^i \left( \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, \hat{x}_t^j) + \varphi(x_t^i) - \varphi(\hat{x}_t^i) \right) \right) dt.$$

By Assumption (A4),  $\bar{x}_t^i(\varphi(x_t^i) - \varphi(\hat{x}_t^i)) \leq 0$ , and

$$\begin{aligned} |\bar{x}_T^i|^2 &= \int_0^T 2\bar{x}_t^i \left( \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, \hat{x}_t^j) \right) dt + \int_0^T 2\bar{x}_t^i (\varphi(x_t^i) - \varphi(\hat{x}_t^i)) dt \\ &\leq \int_0^T 2|\bar{x}_t^i| \left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, \hat{x}_t^j) \right| dt \\ &\leq \int_0^T 2|\bar{x}_t^i| \left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) + \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, \hat{x}_t^j) \right| dt \\ &\leq \int_0^T 2|\bar{x}_t^i| \left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right| dt + \int_0^T 2|\bar{x}_t^i| \left| \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, \hat{x}_t^j) \right| dt \\ &\leq \int_0^T 2|\bar{x}_t^i| \left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right| dt + \int_0^T 2|\bar{x}_t^i| Lip(b_0) \frac{1}{N} \sum_{j=1}^N |x_t^j - \hat{x}_t^j| dt \\ &\leq \int_0^T |\bar{x}_t^i|^2 dt + \int_0^T \left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right|^2 dt \\ &\quad + \int_0^T |\bar{x}_t^i|^2 dt + \int_0^T Lip(b_0)^2 \left( \frac{1}{N} \sum_{j=1}^N |x_t^j - \hat{x}_t^j| \right)^2 dt \\ &\leq \int_0^T 2|\bar{x}_t^i|^2 dt + \int_0^T \left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right|^2 dt + \int_0^T Lip(b_0)^2 \frac{1}{N} \sum_{j=1}^N |x_t^j - \hat{x}_t^j|^2 dt. \end{aligned}$$

Hence,

$$|\bar{x}_T^i|^2 \leq \int_0^T 2|\bar{x}_t^i|^2 dt + \int_0^T \left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right|^2 dt + \int_0^T Lip(b_0)^2 |\bar{x}_t^i|^2 dt,$$

and

$$\begin{aligned} &\left| \int b_0(x_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right|^2 \\ &\leq 2 \left| \int |b_0(x_t^i, y) - b_0(\hat{x}_t^i, y)| \mu_t(dy) \right|^2 + 2 \left| \int b_0(\hat{x}_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right|^2 \\ &\leq 2 \left| \int Lip(b_0) |x_t^i - \hat{x}_t^i| \mu_t(dy) \right|^2 + 2 \left| \int b_0(\hat{x}_t^i, y) \mu_t(dy) - \frac{1}{N} \sum_{j=1}^N b_0(\hat{x}_t^i, x_t^j) \right|^2 \\ &= 2Lip(b_0)^2 |\bar{x}_t^i|^2 + \epsilon^2, \end{aligned}$$

with  $\epsilon = O(\frac{1}{\sqrt{N}})$ . Consequently,

$$|\bar{x}_T^i|^2 \leq \int_0^T (2 + \text{Lip}(b_0)^2) |\bar{x}_t^i|^2 ds + \int_0^T (2\text{Lip}(b_0)^2 |\bar{x}_t^i|^2 + \epsilon^2) dt.$$

By the Gronwall's inequality,

$$\begin{aligned} \sup_{0 \leq t \leq T} |\bar{x}_t^i|^2 &\leq \int_0^T \epsilon^2 dt \cdot \exp\left(\int_0^T (2 + 3\text{Lip}(b_0)^2) dt\right) \\ &= \int_0^T \epsilon^2 dt \cdot e^{(2+3\text{Lip}(b_0)^2)T} = O\left(\frac{1}{N}\right). \end{aligned}$$

Therefore,  $\sup_{0 \leq t \leq T} |x_t^i - \hat{x}_t^i|^2 = O\left(\frac{1}{N}\right)$ . □

Second, suppose that the first player chooses a different control function  $\varphi'$  and all other players  $i = 2, 3, \dots, N$  choose to stay with the control function  $\varphi$ , then the corresponding dynamics for the MFG is

$$dx_t^{1'} = \left( \int b_0(x_t^{1'}, y) \mu_t(dy) + \varphi'(x_t^{1'}) \right) dt + \sigma dW_t^1,$$

and the corresponding dynamics for  $N$ -player game are

$$\begin{aligned} d\tilde{x}_t^1 &= \left( \frac{1}{N} \sum_{j=1}^N b_0(\tilde{x}_t^1, \tilde{x}_t^j) + \varphi'(\tilde{x}_t^1) \right) dt + \sigma dW_t^1, \\ d\tilde{x}_t^i &= \left( \frac{1}{N} \sum_{j=1}^N b_0(\tilde{x}_t^i, \tilde{x}_t^j) + \varphi(\tilde{x}_t^i) \right) dt + \sigma dW_t^i, \quad 2 \leq i \leq N. \end{aligned}$$

We can show

**Lemma 3.**  $\sup_{2 \leq i \leq N} \sup_{0 \leq t \leq T} |\hat{x}_t^i - \tilde{x}_t^i| \leq O(\frac{1}{\sqrt{N}})$  if we consider  $\tilde{x}^1, \hat{x}^1$  as additional quantities.

*Proof.* For any  $2 \leq i \leq N$ ,

$$d(\hat{x}_t^i - \tilde{x}_t^i) = \left[ \frac{1}{N} \sum_{j=1}^N (b_0(\hat{x}_t^i, \hat{x}_t^j) - b_0(\tilde{x}_t^i, \tilde{x}_t^j)) + \varphi(\hat{x}_t^i) - \varphi(\tilde{x}_t^i) \right] dt.$$

By Assumption (A4),

$$\begin{aligned}
|\hat{x}_T^i - \tilde{x}_T^i|^2 &= \int_0^T \left[ 2(\hat{x}_t^i - \tilde{x}_t^i) \left( \frac{1}{N} \sum_{j=1}^N (b_0(\hat{x}_t^i, \hat{x}_t^j) - b_0(\tilde{x}_t^i, \tilde{x}_t^j)) + \varphi(\hat{x}_t^i) - \varphi(\tilde{x}_t^i) \right) \right] dt \\
&\leq \int_0^T \left[ 2(\hat{x}_t^i - \tilde{x}_t^i) \left( \frac{1}{N} \sum_{j=1}^N (b_0(\hat{x}_t^i, \hat{x}_t^j) - b_0(\tilde{x}_t^i, \tilde{x}_t^j)) \right) \right] dt \\
&\leq \int_0^T \left[ 2(\hat{x}_t^i - \tilde{x}_t^i) \left( \frac{1}{N} \sum_{j=1}^N Lip(b_0) (|\hat{x}_t^i - \tilde{x}_t^i| + |\hat{x}_t^j - \tilde{x}_t^j|) \right) \right] dt \\
&\leq 2Lip(b_0) \int_0^T \left[ |\hat{x}_t^i - \tilde{x}_t^i| \left( |\hat{x}_t^i - \tilde{x}_t^i| + \frac{1}{N} \sum_{j=1}^N |\hat{x}_t^j - \tilde{x}_t^j| \right) \right] dt \\
&\leq 2Lip(b_0) \int_0^T \left[ |\hat{x}_t^i - \tilde{x}_t^i|^2 + |\hat{x}_t^i - \tilde{x}_t^i| \left( \frac{1}{N} \sum_{j=1}^N |\hat{x}_t^j - \tilde{x}_t^j| \right) \right] dt \\
&\leq 2Lip(b_0) \int_0^T \left[ |\hat{x}_t^i - \tilde{x}_t^i|^2 + \frac{1}{2N} \sum_{j=1}^N (|\hat{x}_t^i - \tilde{x}_t^i|^2 + |\hat{x}_t^j - \tilde{x}_t^j|^2) \right] dt \\
&\leq 2Lip(b_0) \int_0^T \left[ |\hat{x}_t^i - \tilde{x}_t^i|^2 + \frac{1}{2N} \sum_{j=1}^N (|\hat{x}_t^i - \tilde{x}_t^i|^2 + |\hat{x}_t^j - \tilde{x}_t^j|^2) \right] dt \\
&\leq Lip(b_0) \int_0^T \left[ 3|\hat{x}_t^i - \tilde{x}_t^i|^2 + \frac{1}{N} \sum_{j=1}^N |\hat{x}_t^j - \tilde{x}_t^j|^2 \right] dt,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{2 \leq i \leq N} \sup_{0 \leq t \leq T} |\hat{x}_t^i - \tilde{x}_t^i|^2 &\leq Lip(b_0) \int_0^T \left[ \sup_{2 \leq i \leq N} \sup_{0 \leq s \leq t} 3|\hat{x}_s^i - \tilde{x}_s^i|^2 \right. \\
&\quad \left. + \frac{N-1}{N} \sup_{2 \leq j \leq N} \sup_{0 \leq s \leq t} |\hat{x}_s^j - \tilde{x}_s^j|^2 + \frac{1}{N} |\hat{x}_t^1 - \tilde{x}_t^1|^2 \right] dt \\
&= Lip(b_0) \int_0^T \left[ \frac{4N-1}{N} \sup_{2 \leq i \leq N} \sup_{0 \leq s \leq t} |\hat{x}_s^i - \tilde{x}_s^i|^2 + \frac{1}{N} |\hat{x}_t^1 - \tilde{x}_t^1|^2 \right] dt.
\end{aligned}$$

By the Gronwall's inequality,

$$\sup_{2 \leq i \leq N} \sup_{0 \leq t \leq T} |\hat{x}_t^i - \tilde{x}_t^i|^2 \leq Lip(b_0) \int_0^T \frac{1}{N} |\hat{x}_t^1 - \tilde{x}_t^1|^2 dt \cdot e^{\int_0^T Lip(b_0) \frac{4N-1}{N} dt} = O\left(\frac{1}{N}\right).$$

So,  $\sup_{2 \leq i \leq N} \sup_{0 \leq t \leq T} |\hat{x}_t^i - \tilde{x}_t^i| = O(\frac{1}{\sqrt{N}})$ .  $\square$

From the Lemma 2, for any  $2 \leq i \leq N$ ,  $\sup_{0 \leq t \leq T} |x_t^i - \hat{x}_t^i| = O\left(\frac{1}{\sqrt{N}}\right)$ . Then, by the triangle inequality,  $\sup_{2 \leq i \leq N} \sup_{0 \leq t \leq T} |x_t^i - \tilde{x}_t^i| = O(\frac{1}{\sqrt{N}})$ . Therefore,

$$\sup_{2 \leq i \leq N} \sup_{0 \leq t \leq T} |x_t^i - \tilde{x}_t^i| + \sup_{1 \leq i \leq N} \sup_{0 \leq t \leq T} |x_t^i - \hat{x}_t^i| = O\left(\frac{1}{\sqrt{N}}\right).$$



Third and last, define

$$dx_t^{1''} = \left( \frac{1}{N} \sum_{j=1}^N b_0(x_t^{1''}, x_t^j) + \varphi'(x_t^{1''}) \right) dt + \sigma dW_t^1.$$

Since  $(x - y)(\varphi'(x) - \varphi'(y)) \leq 0$ , then an approach as in Lemma 2 shows

$\sup_{0 \leq t \leq T} |x_t^{1''} - \tilde{x}_t^1| = O\left(\frac{1}{\sqrt{N}}\right)$  and  $\sup_{0 \leq t \leq T} |x_t^{1''} - x_t^{1'}| = O\left(\frac{1}{\sqrt{N}}\right)$ . Therefore,

$$\begin{aligned} J_N^1(s, x^1; \varphi', \varphi) &= E \left[ \int_s^T \frac{1}{N} \sum_{j=1}^N f_0(\tilde{x}_t^1, \tilde{x}_t^j) dt + g(\tilde{x}_t^1) \varphi'(\tilde{x}_t^1) dt \right] \\ &\geq E \left[ \int_s^T \frac{1}{N} \sum_{j=1}^N f_0(\tilde{x}_t^1, x_t^j) dt + g(\tilde{x}_t^1) \varphi'(\tilde{x}_t^1) dt \right] - O\left(\frac{1}{\sqrt{N}}\right) \\ &\geq E \left[ \int_s^T \frac{1}{N} \sum_{j=1}^N f_0(x_t^{1''}, x_t^j) dt + g(x_t^{1''}) \varphi'(x_t^{1''}) dt \right] - O\left(\frac{1}{\sqrt{N}}\right) \\ &\geq E \left[ \int_s^T \int f_0(x_t^{1'}, y) \mu_t(dy) + g(x_t^{1'}) \varphi'(x_t^{1'}) dt \right] - O\left(\frac{1}{\sqrt{N}}\right) \\ &\geq E \left[ \int_s^T \int f_0(x_t^1, y) \mu_t(dy) + g(x_t^1) \varphi(x_t^1) dt \right] - O\left(\frac{1}{\sqrt{N}}\right) \\ &= J_N^1(s, x^1; \varphi, \varphi) - O\left(\frac{1}{\sqrt{N}}\right), \end{aligned}$$

the last inequality is due to the optimality of  $\varphi$  as the optimal control function of the mean field game (5)). This completes the proof.

## 4 An MFG with Singular Controls

In this section, we study a particular MFG and provide explicit solutions. For comparison purposes, we present a singular control counterpart of the MFG originally formulated with regular controls by [17] for systemic risk. We will see that our solution structure is consistent with theirs, despite the differences in problem settings.

The basic idea behind the interbank systemic risk model of [17] is as follows. (A similar model can also be found in [22].) There are  $N$  banks in the system that borrow and lend money among each other. Each bank controls its rates of borrowing and lending to minimize a cost function. There are common noise and individual noise for each bank. Define  $x_t^i$  to be the log-monetary reserve for bank  $i$  with  $i = 1, 2, \dots, N$ . Then, the dynamics of  $x_t^i$  is assumed to be

$$\begin{aligned} dx_t^i &= \frac{a}{N} \sum_{j=1}^N (x_t^j - x_t^i) dt + \xi_t^i dt + \sigma(\rho dW_t^0 + \sqrt{1 - \rho^2} dW_t^i), \\ &= a(m_t - x_t^i) dt + \xi_t^i dt + \sigma(\rho dW_t^0 + \sqrt{1 - \rho^2} dW_t^i), \quad x_s^i = x^i. \end{aligned} \tag{14}$$

Here,  $\{W_t^i\}_{0 \leq t \leq T}$  represents the individual noise for the  $i$ th player with  $i = 1, 2, \dots, N$ , and  $\{W_t^0\}_{0 \leq t \leq T}$  is another independent Brownian motion representing the common noise,  $m_t = \frac{1}{N} \sum_{j=1}^N x_t^j$  with  $m_s = m$ ,  $\xi_t^i$  is the control by bank  $i$ ,  $a$  is a mean-reversion rate, and  $\sigma$ ,  $\rho$ ,  $q$ ,  $c$ , and  $\epsilon$  are nonnegative constants.

The objective is to solve this stochastic game over an admissible control set  $\mathcal{A}$ , which includes adapted processes satisfying proper integrability condition. That is to solve

$$v^i(s, x^i, m) = \inf_{\xi^i \in \mathcal{A}} E_{s, x^i, m} \left[ \int_s^T \left( \frac{1}{2} (\xi_t^i)^2 - q \xi_t^i (m_t - x_t^i) + \frac{\epsilon}{2} (m_t - x_t^i)^2 \right) dt + \frac{c}{2} (m_T - x_T^i)^2 \right], \quad (15)$$

subject to Eqn. (14).

For this MFG, [17] derives a solution as  $N \rightarrow \infty$  with a mean information process defined as  $m_t = \int x \mu_t(dx)$ , where  $\mu_t$  is a probability measure of  $x_t^i$ . The MFG is shown to have a unique optimal control  $\xi_t^{i*}$ , with its mean information process  $m_t^*$  and its value function  $v^i$  given by

$$\begin{aligned} dm_t^* &= \rho \sigma dW_t^0, \\ \xi_t^{i*}(x^i, m) &= q(m - x^i) - \partial_x v^i, \quad \forall i, \\ v^i(s, x^i, m) &= \frac{F_s^1}{2} (m - x^i)^2 + F_s^2, \quad \forall i, \end{aligned} \quad (16)$$

for some deterministic functions  $F_s^1$  and  $F_s^2$ .

**Our model.** If one relaxes the technical assumption of absolute continuity for the bank reserve level  $x_t^i$ , then the processes  $x_t^i$  are finite variation processes. Furthermore, assuming (realistically) that the rate of bank borrowing and lending is bounded, the MFG takes the following form

$$v^i(s, x^i, m) = \inf_{\xi^i \in \mathcal{U}} E_{s, x^i, m} \left[ \int_s^T (r |\dot{\xi}_t^i| + \frac{\epsilon}{2} (m_t - x_t^i)^2) dt + \frac{c}{2} (m_T - x_T^i)^2 \right], \quad (17)$$

subject to

$$\begin{aligned} dx_t^i &= a(m_t - x_t^i) dt + d\xi_t^i + \sigma(\rho dW_t^0 + \sqrt{1 - \rho^2} dW_t^i), \\ &= \left[ a(m_t - x_t^i) + \dot{\xi}_t^i \right] dt + \sigma(\rho dW_t^0 + \sqrt{1 - \rho^2} dW_t^i), \quad x_s^i = x^i, m_s = m. \end{aligned}$$

Here  $m_t = \int x \mu_t(dx)$  is a mean information process with  $\mu_t$  a probability measure of  $x_t^i$ .  $r, \epsilon, c, a, \sigma, \rho$  are nonnegative constants, and the admissible control set is given by

$$\mathcal{U} = \{ \{\dot{\xi}_t^i\} \mid \{\xi_t^i\} \text{ is } \mathcal{F}_t\text{-progressively measurable, finite variation, } \xi_0 = 0, \dot{\xi}_t^i \in [-\theta, \theta] \}.$$

**Remark 1.** It is worth noting that the choice of  $q = 0$  here is mainly for exposition simplicity and does not change the general solution structure. Also, instead of the quadratic form,  $|\dot{\xi}_t^i|$  is used. Technically, it could be replaced by any convex and symmetric function as far as explicit solution is concerned, as demonstrated in Karatzas [31] which generalizes the earlier work of [5] for singular control problems.

This particular MFG appears different from the general problem setting presented in Eqn. (1), with the additional term of common noise. We will show, nevertheless, that appropriate conditioning argument coupled with the symmetric structure in the problem will reduce this MFG to the case without common noise.

There are alternative methods to deal with the presence of common noise [15, 17]. For instance, one could consider the Kolmogorov forward equation and the HJB equation with common noise in their perspective stochastic partial differential equation forms as

$$d\mu_t = \left[ -\partial_x \left( (a(m_t - x) + \dot{\xi}_t^{i*}) \mu_t \right) + \frac{1}{2} \sigma^2 (1 - \rho^2) \partial_{xx} \mu_t \right] dt - \rho \sigma \partial_x \mu_t dW_t^0,$$

where  $\mu_t$  is the probability measure of  $x_t$ , and

$$\begin{aligned} dv^i &+ \left[ \frac{1}{2} \sigma^2 (1 - \rho^2) \partial_{xx} v^i + \mathcal{L}^m v^i + \partial_{xm} v^i \frac{d \langle m, x \rangle}{dt} \right] dt \\ &+ \inf_{\xi \in \mathcal{U}} \left[ (a(m - x) + \dot{\xi}) \partial_x v^i + r |\dot{\xi}| + \frac{\epsilon}{2} (m - x)^2 \right] dt \\ &+ \rho \sigma \partial_m v^i dW_t^0 + \rho \sigma \partial_x v^i dW_t^0 = 0, \end{aligned} \quad (18)$$

for the value function  $v^i(s, x, m)$ , where  $\mathcal{L}^m + \rho \sigma \partial_m dW_t^0$  is an infinitesimal generator for  $m_t$ .

**Solution for  $\rho = 0$  (no common noise).** Now, the Kolmogorov forward equation and the HJB equation are

$$\partial_t \mu_t = -\partial_x \left( (a(m_t - x) + \dot{\xi}_t^{i*}) \mu_t \right) + \frac{1}{2} \sigma^2 \partial_{xx} \mu_t,$$

where  $\mu_t$  is the probability distribution for  $x_t^i$ , and

$$\partial_t v^i + \frac{1}{2} \sigma^2 \partial_{xx} v^i + \inf_{\xi \in \mathcal{U}} \{ (a(m - x) + \dot{\xi}) \partial_x v^i + r |\dot{\xi}| + \frac{\epsilon}{2} (m - x)^2 \} = 0,$$

with the terminal condition  $v^i(T, x, m) = \frac{\epsilon}{2} (m - x)^2$ .

First, fix  $m_t$  as a deterministic process with  $m_s = m \in \mathbb{R}$ . The problem now is a stochastic control problem. According to Eqn. (17), for any  $s \in [0, T]$ , the value function  $v^i(s, \cdot, m)$  for the stochastic control problem is symmetric with respect to  $m$ . That is,  $v^i(s, m - h, m) = v^i(s, m + h, m)$  for any  $h \geq 0$ . By Proposition 1, the optimal control is simply

$$\dot{\xi}_t^{i*}(x, m) = \begin{cases} \theta & \text{if } \partial_x v^i \leq -r, \\ 0 & \text{if } -r < \partial_x v^i < r, \\ -\theta & \text{if } r \leq \partial_x v^i. \end{cases}$$

Note that the associated HJB for the value function with a fixed control is

$$\partial_t v^i + \frac{\epsilon}{2} (m - x)^2 + a(m - x) \partial_x v^i + \frac{1}{2} \sigma^2 \partial_{xx} v^i + \theta \min\{0, r + \partial_x v^i, r - \partial_x v^i\} = 0, \quad (19)$$

with the terminal condition  $v^i(T, x, m) = \frac{\epsilon}{2} (m - x)^2$ .

One can solve for the value function explicitly. Indeed, since the value function  $v(s, \cdot, m)$  is convex, define

$$f_1(s, m) = \sup\{x : \partial_x v^i(s, x, m) = -r\},$$

and

$$f_2(s, m) = \inf\{x : \partial_x v^i(s, x, m) = r\}.$$

Then, on  $f_1(s, m) \leq x \leq f_2(s, m)$ ,

$$\partial_t v^i + \frac{\epsilon}{2}(m-x)^2 + a(m-x)\partial_x v^i + \frac{1}{2}\sigma^2 \partial_{xx} v^i = 0, \quad v^i(T, m, x) = \frac{c}{2}(m-x)^2. \quad (20)$$

The Laplace transform of  $v^i$  is  $\tilde{v}^i(\lambda, x, m) = \int_{-\infty}^T e^{-\lambda t} \partial_t v^i(t, x, m) dt$  for  $\lambda < 0$ , which satisfies

$$\begin{aligned} \widetilde{\partial_t v^i}(\lambda, x, m) &= \int_{-\infty}^T e^{-\lambda t} \partial_t v^i(t, x, m) dt \\ &= e^{-\lambda T} v^i(T, x, m) + \lambda \int_{-\infty}^T e^{-\lambda t} v^i(t, x, m) dt \\ &= e^{-\lambda T} \frac{\epsilon}{2}(m-x)^2 + \lambda \int_{-\infty}^T e^{-\lambda t} v^i(t, x, m) dt \\ &= e^{-\lambda T} \frac{\epsilon}{2}(m-x)^2 + \lambda \tilde{v}^i(\lambda, x, m). \end{aligned}$$

Thus,

$$\left(1 - \frac{1}{\lambda}\right) \frac{\epsilon}{2} e^{-\lambda T} (m-x)^2 + \lambda \tilde{v}^i + a(m-x) \partial_x \tilde{v} + \frac{1}{2} \sigma^2 \partial_{xx} \tilde{v} = 0. \quad (21)$$

A particular solution to (21) is given by

$$\frac{\epsilon}{4a-2\lambda} \left(1 - \frac{1}{\lambda}\right) e^{-\lambda T} (m-x)^2 - \frac{\sigma^2 \epsilon}{\lambda(4a-2\lambda)} \left(1 - \frac{1}{\lambda}\right) e^{-\lambda T}.$$

The fundamental solutions to (21) are sums of two parabolic cylinder functions

$$\begin{aligned} \tilde{\phi}_1(\lambda, m-x) &= e^{\frac{a(x-m)^2}{2\sigma^2}} D_{\frac{\lambda}{a}} \left( -\frac{x-m}{\sigma} \sqrt{2a} \right), \\ \tilde{\psi}_1(\lambda, m-x) &= e^{\frac{a(x-m)^2}{2\sigma^2}} D_{\frac{\lambda}{a}} \left( \frac{x-m}{\sigma} \sqrt{2a} \right) = \tilde{\phi}_1(\lambda, x-m), \end{aligned}$$

where  $D_\alpha(x) = \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\frac{t^2}{2}-xt} dt$ . Therefore, solution to (21) is

$$\begin{aligned} \tilde{v}^i(\lambda, x, m) &= \frac{\epsilon}{4a-2\lambda} \left(1 - \frac{1}{\lambda}\right) e^{-\lambda T} (m-x)^2 - \frac{\sigma^2 \epsilon}{\lambda(4a-2\lambda)} \left(1 - \frac{1}{\lambda}\right) e^{-\lambda T} \\ &\quad + c_1 \tilde{\phi}_1(\lambda, m-x) + c_2 \tilde{\phi}_1(\lambda, x-m) \\ &= \frac{\epsilon}{4a-2\lambda} \left(1 - \frac{1}{\lambda}\right) e^{-\lambda T} (m-x)^2 - \frac{\sigma^2 \epsilon}{\lambda(4a-2\lambda)} \left(1 - \frac{1}{\lambda}\right) e^{-\lambda T} \\ &\quad + \frac{c_1}{\Gamma(-\frac{\lambda}{a})} \int_0^\infty z^{-\frac{\lambda}{a}-1} e^{-\frac{z^2}{2} + \frac{x-m}{\sigma} \sqrt{2a} z} dz + \frac{c_2}{\Gamma(-\frac{\lambda}{a})} \int_0^\infty z^{-\frac{\lambda}{a}-1} e^{-\frac{z^2}{2} - \frac{x-m}{\sigma} \sqrt{2a} z} dz. \end{aligned}$$

for some constant  $c_1, c_2$ . That is,

$$\tilde{v}^i(\lambda, x, m) = \tilde{\eta}_1(\lambda)(m-x)^2 + \tilde{\eta}_3(\lambda) + c_1\tilde{\phi}_1(\lambda, m-x) + c_2\tilde{\phi}_1(\lambda, x-m).$$

Inverting this function yields the solution to the original PDE (20),

$$v^i(s, x, m) = \eta_1(s)(m-x)^2 + \eta_3(s) + c_1\phi_1(s, m-x) + c_2\phi_1(s, x-m).$$

where  $\phi_1$  is the inverse Laplace transform of  $\tilde{\phi}_1$ . Here  $\eta_1(s), \eta_2(s), \eta_3(s)$  are solutions to ODEs

$$\begin{aligned} \partial_t \eta_1 - 2a\eta_1 + \frac{\epsilon}{2} &= 0, & \eta_1(T) &= \frac{c}{2}, \\ \partial_t \eta_3 + \sigma^2 \eta_1 &= 0, & \eta_3(T) &= 0, \end{aligned}$$

and can be expressed explicitly as

$$\eta_1(s) = \left(\frac{c}{2} - \frac{\epsilon}{4a}\right) e^{2a(s-T)} + \frac{\epsilon}{4a},$$

and

$$\eta_3(s) = -\sigma^2 \frac{1}{2a} \left(\frac{c}{2} - \frac{\epsilon}{4a}\right) e^{2a(s-T)} - \sigma^2 \frac{\epsilon}{4a} (s-T) + \sigma^2 \frac{1}{2a} \left(\frac{c}{2} - \frac{\epsilon}{4a}\right).$$

Similarly, on  $x < f_1(s, m)$ , the HJB equation is

$$\partial_t v^i + \frac{\epsilon}{2}(m-x)^2 + r\theta + (a(m-x) + \theta)\partial_x v^i + \frac{1}{2}\sigma^2 \partial_{xx} v^i = 0, \quad v^i(T, m, x) = \frac{c}{2}(m-x)^2.$$

The solution is

$$v^i(s, x, m) = \zeta_1(s)(m-x)^2 + \zeta_2(s)(m-x) + \zeta_3(s) + c_3\phi_2(s, m-x) + c_4\psi_2(s, m-x),$$

where  $\phi_2, \psi_2$  are the inverse Laplace transforms of  $\tilde{\phi}_2$  and  $\tilde{\psi}_2$  respectively, with

$$\tilde{\phi}_2(\lambda, m-x) = e^{\frac{a(x-m-\frac{\theta}{a})^2}{2\sigma^2}} D_{\frac{\lambda}{a}} \left( -\frac{x-m-\frac{\theta}{a}}{\sigma} \sqrt{2a} \right),$$

and

$$\tilde{\psi}_2(\lambda, m-x) = e^{\frac{a(x-m-\frac{\theta}{a})^2}{2\sigma^2}} D_{\frac{\lambda}{a}} \left( \frac{x-m-\frac{\theta}{a}}{\sigma} \sqrt{2a} \right).$$

Here  $\zeta_1(s), \zeta_2(s), \zeta_3(s)$  satisfy

$$\begin{aligned} \partial_t \zeta_1 - 2a\zeta_1 + \frac{\epsilon}{2} &= 0, & \zeta_1(T) &= \frac{c}{2}, \\ \partial_t \zeta_2 - a\zeta_2 - 2\theta\zeta_1 &= 0, & \zeta_2(T) &= 0, \\ \partial_t \zeta_3 + r\theta - \theta\zeta_2 + \sigma^2 \zeta_1 &= 0, & \zeta_3(T) &= 0. \end{aligned}$$

Hence,

$$\zeta_1(s) = \left(\frac{c}{2} - \frac{\epsilon}{4a}\right) e^{2a(s-T)} + \frac{\epsilon}{4a},$$

$$\zeta_2(s) = -\frac{\theta}{a} \left( c - \frac{\epsilon}{a} \right) e^{a(s-T)} + \frac{\theta}{a} \left( c - \frac{\epsilon}{2a} \right) e^{2a(s-T)} - \frac{\theta\epsilon}{2a^2},$$

and

$$\begin{aligned} \zeta_3(s) = & \left( -r\theta - \frac{\theta^2\epsilon}{2a^2} - \frac{\epsilon\sigma^2}{4a} \right) (s-T) - \frac{\theta^2}{a^2} \left( c - \frac{\epsilon}{a} \right) (e^{a(s-T)} - 1) \\ & + \left( \frac{\theta^2}{2a^2} \left( c - \frac{\epsilon}{2a} \right) - \frac{\sigma^2}{2a} \left( \frac{c}{2} - \frac{\epsilon}{4a} \right) \right) (e^{2a(s-T)} - 1). \end{aligned}$$

On  $f_2(s, m) < x$ , the HJB equation is

$$\partial_t v^i + \frac{\epsilon}{2}(m-x)^2 + r\theta + (a(m-x) - \theta)\partial_x v^i + \frac{1}{2}\sigma^2\partial_{xx} v^i = 0, \quad v^i(T, m, x) = \frac{c}{2}(m-x)^2.$$

Therefore,

$$v^i(s, x, m) = \Lambda_1(s)(m-x)^2 + \Lambda_2(s)(m-x) + \Lambda_3(s) + c_5\psi_2(s, x-m) + c_6\phi_2(s, x-m).$$

where  $\Lambda_1, \Lambda_2, \Lambda_3$  satisfy

$$\begin{aligned} \partial_t \Lambda_1 - 2a\Lambda_1 + \frac{\epsilon}{2} &= 0, & \Lambda_1(T) &= \frac{c}{2}, \\ \partial_t \Lambda_2 - a\Lambda_2 + 2\theta\Lambda_1 &= 0, & \Lambda_2(T) &= 0, \\ \partial_t \Lambda_3 + r\theta + \theta\Lambda_2 + \sigma^2\Lambda_1 &= 0, & \Lambda_3(T) &= 0. \end{aligned}$$

That is,

$$\begin{aligned} \Lambda_1(s) &= \left( \frac{c}{2} - \frac{\epsilon}{4a} \right) e^{2a(s-T)} + \frac{\epsilon}{4a}, \\ \Lambda_2(s) &= \frac{\theta}{a} \left( c - \frac{\epsilon}{a} \right) e^{a(s-T)} - \frac{\theta}{a} \left( c - \frac{\epsilon}{2a} \right) e^{2a(s-T)} + \frac{\theta\epsilon}{2a^2}, \end{aligned}$$

and

$$\begin{aligned} \Lambda_3(s) = & \left( -r\theta - \frac{\theta^2\epsilon}{2a^2} - \frac{\epsilon\sigma^2}{4a} \right) (s-T) - \frac{\theta^2}{a^2} \left( c - \frac{\epsilon}{a} \right) (e^{a(s-T)} - 1) \\ & + \left( \frac{\theta^2}{2a^2} \left( c - \frac{\epsilon}{2a} \right) - \frac{\sigma^2}{2a} \left( \frac{c}{2} - \frac{\epsilon}{4a} \right) \right) (e^{2a(s-T)} - 1). \end{aligned}$$

Note that  $\eta_1(s) = \zeta_1(s) = \Lambda_1(s)$ ,  $-\zeta_2(s) = \Lambda_2(s)$ , and  $\zeta_3(s) = \Lambda_3(s)$ . Hence, the value function  $v$  is also symmetric to  $m$ , meaning  $c_1 = c_2$ ,  $c_3 = c_6$  and  $c_4 = c_5$ . Moreover, the regularity and convexity of  $v(s, \cdot, m)$  implies that  $\partial_x v$  is nondecreasing and that there are  $x_1 < x_2$  satisfying

$$x_1 = \sup\{x : \partial_x v(s, x, m) = -r\},$$

and

$$x_2 = \inf\{x : \partial_x v(s, x, m) = r\}.$$

Now, the symmetry of  $v(s, \cdot, m)$  with respect to  $m$  implies that  $x_1 = m - h$  and  $x_2 = m + h$  for some  $h > 0$ .

In fact, one can solve for  $c_1, c_3, c_4$  and  $h$  from the  $\mathcal{C}^2$ -smoothness of  $v^i(s, \cdot, m)$  at  $x_2$ , and get

$$\begin{aligned}
c_1 &= \frac{2\Lambda_1(s)h - r}{-\phi'_1(s, h) + \phi'_1(s, -h)}, \\
c_3 &= \frac{1}{\phi_2(s, -h)} \left( (2\Lambda_1(s)h - r) \frac{\phi_1(s, h) + \phi_1(s, -h)}{-\phi'_1(s, h) + \phi'_1(s, -h)} - \Lambda_2(s)h - \Lambda_3(s) + \eta_2(s) \right) \\
&\quad - \frac{\phi_2(s, -h)}{\psi_2(s, -h)} \frac{1}{\psi'_2(s, -h)\phi_2(s, -h) - \psi_2(s, -h)\phi'_2(s, -h)} (\phi_2(s, -h)(2\Lambda_1(s)h + \Lambda_2(s) - r) \\
&\quad - \phi'_2(s, -h)((2\Lambda_1(s)h - r) \frac{\phi_1(s, h) + \phi_1(s, -h)}{-\phi'_1(s, h) + \phi'_1(s, -h)} - \Lambda_2(s)h - \Lambda_3(s) + \eta_2(s))), \\
c_4 &= \frac{1}{\psi'_2(s, -h)\phi_2(s, -h) - \psi_2(s, -h)\phi'_2(s, -h)} (\phi_2(s, -h)(2\Lambda_1(s)h + \Lambda_2(s) - r) \\
&\quad - \phi'_2(s, -h)((2\Lambda_1(s)h - r) \frac{\phi_1(s, h) + \phi_1(s, -h)}{-\phi'_1(s, h) + \phi'_1(s, -h)} - \Lambda_2(s)h - \Lambda_3(s) + \eta_2(s))),
\end{aligned}$$

and eventually

$$c_4\psi''_2(s, -h) + c_3\phi''_2(s, -h) = c_1\phi''_1(s, h) + c_1\phi''_1(s, -h).$$

By definitions of  $x_1$  and  $x_2$  and convexity of  $v(s, \cdot, m)$ ,  $x_1 = m - h$  and  $x_2 = m + h$  with

$$h = \inf\{\kappa : c_4\psi''_2(s, -\kappa) + c_3\phi''_2(s, -\kappa) = c_1\phi''_1(s, \kappa) + c_1\phi''_1(s, -\kappa)\}. \quad (22)$$

Note that the degenerate case of  $h = \infty$  means that there is no action region and  $\xi_t^{i*} = 0$  for all  $x \in \mathbb{R}$ .

Second, we can solve the McKean–Vlasov equation. Let  $m'_t$  be an updated mean information process with  $\xi_t^{i*}$  such that  $m'_t = \int x\mu_t(dx)$ , with  $\mu_t$  a probability measure of optimal state process  $x_t^i$  and the controlled dynamics of  $x_t$  given by

$$dx_t^i = \left[ a(m'_t - x) + \dot{\xi}_t^{i*} \right] dt + \sigma dW_t^i, \quad x_s^i = x, m'_s = m.$$

Then, the Kolmogorov forward equation for  $\mu_t$  is

$$\partial_t \mu_t = -\partial_x \left( (a(m'_t - x) + \dot{\xi}_t^{i*}) \mu_t \right) + \frac{1}{2} \sigma^2 \partial_{xx} \mu_t. \quad (23)$$

From the symmetry of  $x_t^i$ ,  $P(x_t^i > x_2 = m + h) = P(x_t^i < x_1 = m - h)$  and

$$\dot{\xi}_t^{i*}(x, m) = \begin{cases} \theta, & \text{if } x \leq x_1 = m - h, \\ 0, & \text{if } x_2 < x < x_1, \\ -\theta, & \text{if } x_2 = m + h \leq x. \end{cases} \quad (24)$$

Then, by (23) and the definition of  $m'_t$ ,

$$dm'_t = \theta(P(x_t^i > x_2) - P(x_t^i < x_1))dt = 0,$$

where  $x_1, x_2$  and  $h$  are given in Eqn. (22). Hence,  $m'_t$  satisfies  $dm'_t = 0$  with  $m'_s = m$  for  $t \in [s, T]$ .

Finally, combine all above steps and note that the updated mean information process from the McKean–Vlasov equation is  $m'_t = m$ , which is independent of the optimal control from the stochastic control step. Hence, for each iteration after the first step, the updated mean information processes remain  $m'_t = m$ , hence it is a fixed point mean information process solution. In summary, the solution to the MFG (17) with  $\rho = 0$  is given by Eqn. (24) for the optimal control, and

$$dm_t^* = 0 \quad \forall t \in [s, T], \quad m_s^* = m, \quad (25)$$

$$\begin{aligned} v^i(s, x, m) = & a_1(s)(m - x)^2 + a_2(s)(m - x) + a_3(s) \\ & + a_4(s)\phi_1(s, m - x) + a_5(s)\phi_1(s, x - m) + a_6(s)\phi_2(s, m - x) + a_7(s)\psi_2(s, m - x). \end{aligned} \quad (26)$$

$$\begin{aligned} \text{for deterministic functions } a_j(s) = & \begin{cases} \zeta_j(s), & \text{if } x < m - h, \\ \eta_j(s), & \text{if } m - h \leq x \leq m + h, \\ \zeta_j(s), & \text{if } m + h < x, \end{cases} \quad \text{for } j = 1, 3, \\ a_2(s) = & \begin{cases} \zeta_2(s), & \text{if } x < m - h, \\ 0, & \text{if } m - h \leq x \leq m + h, \\ -\zeta_2(s), & \text{if } m + h < x, \end{cases} \\ a_j(s) = & \begin{cases} 0, & \text{if } x < m - h, \\ c_1, & \text{if } m - h \leq x \leq m + h, \\ 0, & \text{if } m + h < x, \end{cases} \quad \text{for } j = 4, 5, \\ a_6(s) = & \begin{cases} c_3, & \text{if } x < m - h, \\ 0, & \text{if } m - h \leq x \leq m + h, \text{ and} \\ c_3, & \text{if } m + h < x, \end{cases} \\ a_7(s) = & \begin{cases} c_4, & \text{if } x < m - h, \\ 0, & \text{if } m - h \leq x \leq m + h, \\ c_4, & \text{if } m + h < x. \end{cases} \end{aligned}$$

**Solution for  $\rho \neq 0$  (with common noise).** Similar to the special case without common noise, the value function with a fixed  $m_t$ ,  $v^i(s, \cdot, m)$  is also symmetric with respect to  $m$ . That is,  $v^i(s, m - h, m) = v^i(s, m + h, m)$  for any  $h \geq 0$ . Again by Proposition 1 and from the HJB (18), the optimal control is given by Eqn. (24), which is independent of the common noise, and  $P(x_t^i > x_2 = m + h) = P(x_t^i < x_1 = m - h)$ . Thus, the corresponding Kolmogorov forward equation for  $\mu_t$  given  $\dot{u}_t^{i*}$  is

$$d\mu_t = \left[ -\partial_x \left( (a(m'_t - x) + \dot{\xi}_t^{i*}) \mu_t \right) + \frac{1}{2} \sigma^2 (1 - \rho^2) \partial_{xx} \mu_t \right] dt - \rho \sigma \partial_x \mu_t dW_t^0, \quad (27)$$

and from (27),

$$dm'_t = \theta(P(x_t^i > x_2) - P(x_t^i < x_1))dt + \rho \sigma dW_t^0 = \rho \sigma dW_t^0,$$

where  $x_1, x_2$  and  $h$  are given by Eqn. (22). Hence, conditioned on  $W^0$ ,  $\mathcal{L}^m v^i = d < m, x > = 0$ , and because of  $v(s, x + \Delta, m + \Delta) = v(s, x, m)$  for any  $\Delta > 0$ ,  $\partial_x v^i = -\partial_m v^i$ . Moreover,



$m'_t$  satisfies  $dm'_t = \rho \sigma dW_t^0$  with  $m'_s = m$  for  $t \in [s, T]$ . As in the previous case, the fixed point mean information process solution is

$$dm_t^* = \rho \sigma dW_t^0 \quad \forall t \in [s, T], \quad m_s^* = m. \quad (28)$$

Therefore, from the above two steps, it is clear that conditioned on  $W^0$ , the HJB equation (18) is the same as the HJB equation (19). Hence, the value functions and the optimal controls are the same with or without the common noise, given by Eqns. (26) and (24) respectively.

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